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CLOSED LOXODROMICS OF THE TORUS.

By PROFESSOR ARNOLD EMCH, University of Colorado.

1. In Vol. VI (1899), pp. 136-139,* of this journal, I have proved the following theorem:†

If the ratio R/r of a torus is a rational fraction, and if a loxodromic of the torus winds m times around the axis (z-axis) and n times around the axial circle ($x^2+y^2=R^2$), then every loxodromic intersecting the first orthogonally closes also and their numbers m_1 and n_1 are related to the corresponding numbers m and n of the first by the equation

$$\frac{n}{m}\cdot\frac{n_1}{m_1}=\frac{R^2-r^2}{r^2},$$

where R is the radius of the axial circle and r that of a meridian-section of the torus, and $R>r$.

If the equatorial plane (plane of the axial circle) be assumed as the xy-plane, u as the angle which the plane of a meridian (circle with center C and radius r) makes with xz-plane, and v as the angle which the radius PC from C to a point P of this meridian makes with the perpendicular to the xy-plane, the rectangular coördinates of P may be expressed by

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* Slight corrections appear on p. 188, where a superfluous = occurs. ED. D.

† A torus, anchor-ring, is a surface of revolution generated by a circle which rotates about a fixed axis of its plane. A loxodromic on the torus is a curve which intersects all parallels (and consequently all meridians) at constant angles.

A clear discussion of the torus and its conformal representation may be found in F. Klein's little book: "Ueber Riemann s Theorie der Algebraischen Funktionen und ihrer Integrale." Leipzig, 1882, pp. 50-55.

$$x=(R+r\sin v)\cos u,$$
$$y=(R+r\sin v)\sin u,$$
$$z=r\cos v,$$

and the (u, v)-equation of a closed loxodromic of the prescribed kind by

$$\sin\frac{n}{m}u=\frac{r+R\sin v\,[*]}{R+r\sin v}\ldots.(1).$$

In this note I shall discuss the nature of these curves.

2. For this purpose substitute in (1) for $\sin u$ and $\sin v$ their equivalent expressions

$$\sin u=\frac{y}{\sqrt{(x^2+y^2)}},\quad \sin v=\frac{\sqrt{(x^2+y^2)}-R}{r}.$$

In this manner the Cartesian equation of the projection of the closed loxodromic on the (x, y)-plane is obtained. As n and m are positive integers, $\sin\frac{n}{m}u$ may be expressed algebraically in terms of $\sin u$. Hence, (1) becomes an algebraic expression in x and y, and we have the theorem:

The projection of a closed loxodromic of the torus on the xy-plane is an algebraic curve.

As the torus is a surface of the fourth order it follows further:

The loxodromic itself is algebraic.

3. I shall consider in particular the case where the loxodromic turns only once about the z-axis, *i. e.*, where $m=1$. Now

$$\sin nu=a_1\sin u+a_3\sin^3u+\ldots.+a_n\sin^n u,\ \ldots.(2)$$

when $n=2k+1$ is odd; and

$$\sin nu=\cos u(b_1\sin u+b_3\sin^3 u+\ldots.+b_n\sin^{n-1}u),\ \ldots.(3)$$

when $n=2k$ is even.[†]

In the first case, $n=2k+1$, we obtain after setting $\sin u=\frac{y}{\sqrt{(x^2+y^2)}}$,

$$\sin(2k+1)u=\frac{y}{\sqrt{(x^2+y^2)}}\left[a_1+a_3\frac{y^2}{x^2+y^2}+\ldots.\ a_n\frac{y^{2k}}{(x^2+y^2)^k}\right],$$

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[*] A. Emch: Ueber orthogonale Systeme und einige technische Anvendungen, Biel, 1898.

[†] See Jordan: Cours d'Analyse, Vol. 1, p. 238.

$$\text{or, } \sin(2k+1)u = \frac{y}{\sqrt{(x^2+y^2)}\,(x^2+y^2)^k} f(x, y), \quad \ldots.(4)$$

where $f(x, y)$ is a polynomial in x and y of degree $2k$. Substituting (4) and the value of $\sin v$ in (1) we get

$$\frac{ryf(x, y)}{(x^2+y^2)^k} = r^2 - R^2 + R\sqrt{(x^2+y^2)},$$

or when rationalized and rearranged:

$$[ryf(x, y) + (R^2-r^2)(x^2+y^2)^k]^2 - R^2(x^2+y^2)^{2k+1} = 0, \quad \ldots.(5)$$

which, clearly, is a polynomial of degree $4k+2=2n$.

In the second case, $n=2k$, we have

$$\sin 2k = \frac{xy}{(x^2+y^2)}\left[b_1 + b_3\frac{y^2}{x^2+y^2} + \ldots. + b_n\frac{y^{2k-2}}{(x^2+y^2)^{k-1}}\right], \quad \ldots.(6)$$

$$\text{or, } \sin 2k = \frac{xy}{(x^2+y^2)^k} g(x, y), \quad \ldots.(7)$$

where $g(x, y)$ is a polynomial of degree $2k-2$. Substituting again in (1) and proceeding similarly as when we obtained (5), we get

$$[rxyg(x, y) - R(x^2+y^2)^k]^2 - (r^2 - R^2)(x^2+y^2)^{2k-1} = 0, \quad \ldots.(8)$$

which, clearly, is a polynomial of degree $4k=2n$. We have therefore the theorem:

The xy-projection of a closed loxodromic winding n times around the axial circle is an algebraic curve of order 2n. The loxodromic itself is either of order 4n or else is a part of a curve of this order.

4. As an example take $n=1$, so that (5) becomes

$$R^2(x^2+y^2) = (R^2-r^2+ry)^2$$

This represents an ellipse and the corresponding loxodromic is a circle (see *loc. cit.*), and it must therefore be cut out by a double tangent-plane of the torus. This result may be stated in the theorem:

The double-tangent-planes of a torus cut the torus in circles which intersect the parallels and meridians of the torus under constant angles.

The cylinder $R^2(x^2+y^2)=(R^2-r^2+xy)^2$, z=arbitrary, cuts the torus in two circles of this kind, which are equally inclined to the xy-plane, thus making together a degenerated curve of the fourth order. The remainder of the intersection is an imaginary curve of the fourth order (two imaginary conics).

Taking $R=2r$, we get $\frac{n}{m}\frac{n_1}{m_1}=3$, hence $n_1=3$. The orthogonal loxodromic winds 3 times around the axial circle. Its equation is easily found to be

$$R^2(x^2+y^2)^3=[(R^2-r^2)(x^2+y^2)+r(3x^2y-y^3)]^2,$$

and is, as we expect, of the 6th order.

Every closed loxodromic has an n-fold symmetry with respect to meridian-planes. Hence the cylinder with its xy-projection as a base cuts the torus in another loxodromic which is the reflection of the first on the xy-plane. The loxodromic is thus generally of order $4n$.

The University of Colorado, November, 1902.

ON BALL'S HISTORY OF MATHEMATICS.

By DR. G. A. MILLER.

Among the few histories of mathematics in the English language the third edition of Ball's work is the latest and most extensive.* It is therefore natural that this work should find a place in the libraries of many teachers of mathematics in these days of deep interest in the history of science. The following corrections and additions may possibly prove helpful especially to those who do not have the time or opportunity to study Cantor's great work.†

On page 4, Ball says, "The Egyptians and Greeks simplified the problem by reducing a fraction to the sum of several fractions, in each of which the numerator was unity, so that they had to consider only various denominators: the sole exceptions being the fractions $\frac{2}{3}$ and $\frac{3}{4}$. This remained the Greek practice until the sixth century of our era."

Being unable to find any instance where the ancient Egyptians had used the fraction $\frac{3}{4}$ I recently wrote Ball about the matter and received the following reply, "I cannot find my authority (if I had one) for the statement that the Egyptians used the fraction $\frac{3}{4}$. I fear it must have been an error which I have repeated in each edition without verification."

The above quotation from Ball's History is also apt to convey a false impression in regard to the Greek methods of dealing with fractions. While it is true that the Greeks employed unit fractions to a considerable extent, yet they also employed fractions with a general numerator and had a general notation for such fractions. In these respects they differed very widely from the Egyptians, although one might be led to infer the contrary not only from the above quotation but also from the statement on page 76.

On page 13 it is stated that the history of mathematics written by

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* A Short Account of the History of Mathematics. By W. W. R. Ball, The Macmillan Company, New York, 1901, pp. xxiv+527.

† Vorlesungen ueber Geschichte der Mathematik. Von Moritz Cantor, Teubner, Leipzig, 1884-1901.